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Variable-Mesh Difference Equation for the Stream Function in Axially Symmetric Flow

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A finite difference equation is developed for the stream function in cylindrical coordinates with axial symmetry which is applicable to an irregular mesh having different length and radial dimensions In addition, the length and radial dimensions may be varied, and the mesh made finer in any interior region The equation also takes into account an irregular boundary

THE stream function in cylindrical coordinates for the case of axial symmetry is

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0 \qquad (1)$$

A five-point mesh crossing an irregular boundary is used and is shown in Fig 1 The mesh under consideration has spacing of h units in the z direction and k units in the r direction, and α and β are the ratios of the distance to the boundary divided by the mesh distance If the function $\psi(z,r)$ is expanded in a Taylor's series in the r direction, dropping the argument for the derivatives, the following equations result:

$$\psi(z,r+k) = \psi(z,r) + k\frac{\partial\psi}{\partial r} + \frac{k^2}{2!}\frac{\partial^2\psi}{\partial r^2} + \frac{k^3}{3!}\frac{\partial^3\psi}{\partial r^3} + \frac{k^4}{4!}\frac{\partial^4\psi}{\partial r^4} \qquad (2)$$

$$\psi(z,r-\alpha k) = \psi(z,r) - \alpha k\frac{\partial\psi}{\partial r} + \frac{\alpha^2k^2}{2!}\frac{\partial^2\psi}{\partial r^2} - \frac{\alpha^3k^3}{3!}\frac{\partial^3\psi}{\partial r^3} + \frac{\partial^4k^4}{4!}\frac{\partial^4\psi}{\partial r^4} + \qquad (3)$$

If Eq (2) is multiplied by α and the result added to Eq (3),

$$\frac{\partial^2\psi}{\partial r^2} = \frac{2\psi(z,r+k)}{k^2(1+\alpha)} + \frac{2\psi(z,r-\alpha k)}{k^2\alpha(1+\alpha)} - \frac{2\psi(z,r)}{k^2\alpha^2} + (\alpha^2-1)0k + 0k^2 \qquad (4)$$

where $0k$ and $0k^2$ are terms of the order of k and k^2, respectively

If Eq (2) is multiplied by α^2 and the result subtracted from Eq (3), the following is obtained when dividing by r:

$$\frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{\alpha\psi(z,r+k)}{rk(1+\alpha)} - \frac{\psi(z,r-\alpha k)}{rk\alpha(1+\alpha)} + \frac{(1-\alpha)\psi(z,r)}{rk\alpha} + 0k^2 \qquad (5)$$

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In a similar way,

$$\frac{\partial^2\psi}{\partial z^2} = \frac{2\psi(z-h,r)}{h^2(1+\beta)} + \frac{2\psi(z+\beta h,r)}{h^2\beta(1+\beta)} - \frac{2\psi(z,r)}{h^2\beta^2} + (\beta^2-1)0h + 0h^2 \qquad (6)$$

Equations (4–6) can be substituted in Eq (1), and the result is a difference form of the stream function for the point $\psi(z,r)$ in terms of the four surrounding points:

$$\psi(z,r+k)\left(\frac{2}{1+\alpha}\right)\left(1-\frac{\alpha k}{2r}\right) + \psi(z,r-\alpha k)\left(\frac{2}{1+\alpha}\right)\left(\frac{1}{\alpha}\right)\left(1+\frac{k}{2r}\right) + \psi(z+\beta h,r)\left[\frac{2\lambda^2}{\beta(1+\beta)}\right] + \psi(z-h,r)\left(\frac{2\lambda^2}{1+\beta}\right) - 2\psi(z,r)\left[\frac{\lambda^2}{\beta} + \frac{1}{\alpha} - \frac{k(1-\alpha)}{2\alpha r}\right] + (1-\beta^2)0h - (1-\alpha^2)0k + 0h^2 + 0k^2 = 0 \qquad (7)$$

$$\lambda^2 = k^2/h^2,\ 0 < \alpha \leq 1,\ 0 < \beta \leq 1$$

Equation (7) is valid for any mesh point near a boundary It also applies to an interior point where a change in mesh size is introduced This feature is particularly valuable when evaluating the stream function near an abruptly changing boundary For example, evaluating Eq (7) near a change

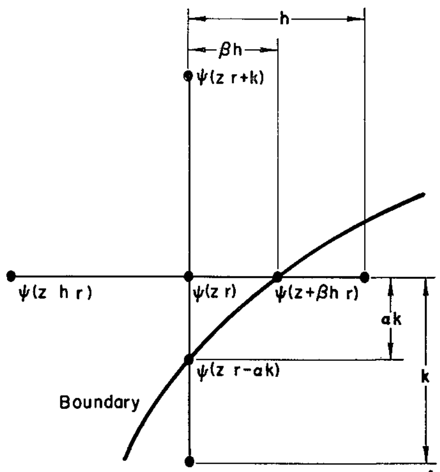

Fig 1 Five-point mesh used in difference equation

in mesh size in the z direction corresponds to a vertical boundary in Fig 1 through the point $\psi(z+h,r)$, and β becomes the ratio of the mesh sizes The mesh need not be square nor regular, that is, k and h need not be equal nor do they always have to be constant

For an interior point in a mesh where $h = k =$ const, Eq. (7) reduces to the familiar form (e g, see Salvadori and Baron[1])

$$\psi(z,r+k)\left(1-\frac{k}{2r}\right) + \psi(z,r-k)\left(1+\frac{k}{2r}\right) + \psi(z-k,r) + \psi(z+k,r) - 4\psi(z,r) = 0 \qquad (8)$$

The error involved is of the order of $(1-\beta^2)h$ in Eq (7) Care must be exercised to ensure that β does not approach zero In constructing the net it is necessary to make $(1-\beta^2) \rightarrow h$ if the whole term is to be of the order of h^2 The same argument holds for α If the net is made fine enough, $h, k \ll 1$, and the characteristic dimension of the body under consideration is unity, the terms $0h^2$ and $0k^2$ tend to zero

Also, it is assumed when using Taylor's expansion that all derivatives are bounded This is, of course, not true at the stagnation point of a body of revolution for example However, if the value of the mesh point at the stagnation point is defined as part of the boundary, the unboundedness problem will be avoided for interior points near the singularity

The stability problem may be discussed along the lines presented by Forsythe and Wasow[2] The term $1 - (\alpha k/2r)$ will be positive if $\alpha < (2r/k)$ If this latter condition is met and if α,β,k,h,r are all positive and finite, Eq (7) merely represents $\psi(z,r)$ as a weighted average of four surrounding points Since the boundary is specified and finite, and since all derivatives are bounded in the region under consideration, all interior points must be finite: $0 \leq m \leq \psi(z,r) \leq M \leq \infty$ Thus, the difference equation should be stable throughout the region interior to the boundary

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Three-Dimensional Boundary Layers with a Normal Wall Velocity

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1 Introduction

Cooke[1] applied the Stewartson[2] transformation, when the crossflow is small, to the three dimensional boundary-layer equations in compressible flow and thereby reduced them to the equations describing a certain three-dimensional incompressible flow In the present note we achieve a similar result for the case where there is a normal velocity at the wall, thus extending the work of the writer[3] to three-dimensional layers with small crossflow

The boundary layer equations when the crossflow is small have been derived by Cooke,[4] using streamline coordinates, in the following form:

$$\rho\left(u\frac{\partial u}{\partial s} + w\frac{\partial u}{\partial \zeta}\right) = \rho_e u_e \frac{\partial u_e}{\partial s} + \frac{\partial}{\partial \zeta}\left(\mu \frac{\partial u}{\partial \zeta}\right) \quad (1)$$

$$\rho\left(u\frac{\partial v}{\partial s} + w\frac{\partial v}{\partial \zeta} + \frac{uv}{r}\frac{\partial r}{\partial s}\right) = K(\rho_e u_e^2 - \rho u^2) + \frac{\partial}{\partial \zeta}\left(\mu\frac{\partial v}{\partial \zeta}\right) \quad (2)$$

$$\rho C_p\left(u\frac{\partial T}{\partial s} + w\frac{\partial T}{\partial \zeta}\right) = -\rho_e u u_e \frac{\partial u_e}{\partial s} + \frac{\partial}{\partial \zeta}\left(k\frac{\partial T}{\partial \zeta}\right) + \mu\left(\frac{\partial u}{\partial \zeta}\right)^2 \quad (3)$$

$$\frac{\partial}{\partial s}(\rho r u) + \frac{\partial}{\partial \zeta}(\rho r w) = 0 \quad (4)$$

The coordinate system (ξ, η, ζ) with corresponding velocity components (u, v, w) is such that ζ is measured normal to the surface (occupying $\zeta = 0$); η = const, $\zeta = 0$ are the projections of the external streamlines on to this surface, and ξ = const, $\zeta = 0$ are their orthogonal trajectories The length element dl is then given by

$$dl^2 = h_1^2(\xi,\eta)d\xi^2 + r^2(\xi,\eta)d\eta^2 + d\zeta^2$$

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Cooke defined ds as the length element along the curves η = const, $\zeta = 0$ Thus, $ds = h_1 d\xi$ and the s and ξ directions are the same In Eqs (1–4), $K = -(\partial h_1/\partial \eta)/h_1 r$ and ρ, T, μ, C_p, and k are the density, temperature, viscosity, specific heat, and thermal conductivity of the fluid The suffix e denotes values just outside the boundary layer, and we shall use suffixes 0 and w for values at a standard isentropic reference position and the wall, respectively

As usual, for correlation we need to make three restrictive assumptions: 1) the Prandtl number σ is unity, so that $k = C_p\mu$; 2) the viscosity is proportional to the temperature although we may allow the proportionality factor to vary with s and η; thus, we choose $\mu = (\mu_w/T_w)T$, where μ_w is related accurately to T_w; and 3) the surface is heat-insulating, $(\partial T/\partial \zeta)_{\zeta=0} = 0$ The boundary conditions for Eqs (1–4) are

$$u = v = 0 \qquad w = w_w(s,\eta) \qquad \partial T/\partial\zeta = 0 \quad \text{at } \zeta = 0$$

$$u = u(s,\eta) \qquad v = 0 \qquad T = T_e(s,\eta) \quad \text{at } \zeta = \infty$$

Under assumptions 1) and 3), the temperature can be written in the form

$$\frac{T}{T_e} = 1 + \frac{(\gamma - 1)}{2a^2}(u_e^2 - u^2) \quad (5)$$

where γ is the ratio of specific heats, and a is the sound speed Equation (5) is the exact solution of Eq (3), consistent with (1) and (4), but is an approximate expression for T since it is assumed that $v \ll u$

2 Analysis

The analysis that follows is similar to the one used by Gribben bearing in mind that now each unknown depends on three independent variables That in turn was based on Illingworth's[5] original form of the correlation transformation which treated the boundary-layer equations written in the Von Mises' coordinates where the stream function is used as an independent variable instead of distance normal to the surface Thus, here, the first step is to transform to new independent variables (x, y, ψ) where

$$x = s \qquad y = \eta \qquad \psi = \psi(s,\eta,\zeta)$$

and ψ is defined to satisfy (4) identically, i e ,

$$\frac{\partial\psi}{\partial\zeta} = \frac{r\rho u}{\rho_0} \qquad \frac{\partial\psi}{\partial s} = -\frac{r\rho w}{\rho_0} \quad (6)$$

In addition, we introduce the nondimensional velocity $\bar{u} = u/u_e$ when Eqs (1) and (2) become

$$\frac{\partial\bar{u}}{\partial x} = \frac{1}{u_e\bar{u}}\frac{\partial u_e}{\partial x}\left(\frac{\rho_e}{\rho} - \bar{u}^2\right) + F(x,y)\frac{\partial}{\partial\psi}\left(\bar{u}\frac{\partial\bar{u}}{\partial\psi}\right) \quad (7)$$

$$\frac{1}{r}\frac{\partial}{\partial x}(vr) = \frac{Ku_e}{\bar{u}}\left(\frac{\rho_e}{\rho} - \bar{u}^2\right) + F(x,y)\frac{\partial}{\partial\psi}\left(\bar{u}\frac{\partial v}{\partial\psi}\right) \quad (8)$$

In these equations the product $\mu\rho$ has been replaced by $\mu_w\rho_w$ in accordance with assumption 2), the perfect gas law, and the fact that the pressure is independent of the ψ coordinate It follows that

$$F(x,y) = \mu_w\rho_w r^2 u_e/\rho_0^2$$

The function ψ is closely related to the stream function of axisymmetric flow but is only determined from (6) to within an arbitrary function of η, say $g(\eta)$ Thus, the value of ψ at the surface is given as

$$(\psi)_{\zeta=0} = f(x,y) = -\int_0^x \frac{r\rho_w w_w}{\rho_0}dx + g(y)$$